Validity in Predicate Logic

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Valid Arguments in Natural Language

- Arguments in natural language consist of a set of sentences serving as **premises** and a single sentence serving as the **conclusion**.
- A natural language argument is valid if and only if it is not possible for all the premises to be true and the conclusion false.
- Validity of natural language arguments can be evaluated by transcribing them into Predicate Logic and applying the semantics to the transcribed arguments.

Valid Arguments in Predicate Logic

- Truth and satisfaction in an interpretation are the most basic semantical properties of sentences of Predicate Logic.
- These properties can be used to determine the truth-value, in an interpretation, of a Predicate Logic sentence (conclusion) relative to a set of Predicate Logic sentences (premises) in an argument of Predicate Logic.
- The goal is to determine whether there is an interpretation in which all the premise-sentences have the value t and the conclusion-sentence has the value f.
- If there is such an interpretation, it is a **counterexample**, and the transcribed argument is **invalid**.
- If there are no counterexamples, then the transcribed argument is **valid**.

Determining Invalidity

- To show that an argument of Predicate Logic is invalid, one produces an interpretation to serve as a counterexample.
- Producing a counterexample requires the specification of a domain, as well as the designations of the names and function symbols, and the extensions of the predicates occurring in the sentence.

- Consider the following argument.
  - Premises: (∃x)Fx, (∃x)Gx
  - Conclusion: (∃x)(Fx & Gx)
- To show the invalidity of this argument, we produce an interpretation which makes the conclusion false, making sure that it allows the premises to be true.

An Example of a Proof of Invalidity

- Let I be an interpretation such that D = {1, 2}, v(F) = {⟨1⟩}, v(G) = {⟨2⟩}
- Let ‘d’ be a variable assignment in I.
- Since ⟨1⟩ ∈ V(F), d[1/x] satisfies ‘Fx,’ so, any d satisfies ‘(∃x)Fx.’
- Since ⟨2⟩ ∈ V(G) d[2/x] satisfies ‘Gx’, so any d satisfies ‘(∃x)Gx.’
- So, both premises are true in I.
- No single x-variant of d satisfies both ‘Fx’ and ‘Gx,’ and so none satisfies ‘Fx & Gx,’ so ‘(∃x)(Fx & Gx)’ is not satisfied by d and is false in I.
- So on this interpretation I, the premises are true and the conclusion false, which demonstrates the invalidity of the argument.

The Example Using Substitutional Semantics

- Let I be the following interpretation: D = {a, b}, Fa & ~Fb & ~Ga & Gb.
- Since ‘Fa’ is true in I, ‘(∃ x)Fx’ is true in I.
- Since ‘Gb’ is true in I, ‘(∃ x)Gx’ is true in I.
- However, since ‘Fb’ and ‘Ga’ are false in I, both ‘Fa & Ga’ and ‘Fb & Gb’ are false in I.
- Therefore, ‘(∃x)(Fx & Gx)’ is false in I.
- So on this interpretation I, the premises are true and the conclusion false, which demonstrates the invalidity of the argument.

Determining Validity

- Because validity of arguments is defined in terms of all possible interpretations, it cannot be proved on the basis of a single interpretation.
- General reasoning about interpretations is required.
- For this reason, we use metavariables to indicate arbitrary:

  - Interpretations **I**
  - Domains **D**
  - Objects in the domain **u** (with or without positive integer subscripts)
  - Valuation functions **v**

- At this level of generality, we can still draw conclusions about what must hold if the premises of an argument are to be true in an arbitrary interpretation.

An Example of a Proof of Validity

- To prove: {(∀x)(Fx ⊃ Gx), Fa} ⊨ Ga
- Let **I** be an arbitrary interpretation, **v** the valuation function in **I**, and **d** an arbitrary variable assignment.
- Suppose that '(∀x)(Fx ⊃ Gx)' and 'Fa' are true in **I**, in which case both sentences are satisfied by all variable assignments.
- So, for all **u** in the domain of **I**, **d**[**u**/x] satisfies 'Fx ⊃ Gx.'
- Suppose further that **v**(a) = $\mathbf{u}_1$, for an arbitrary member $\mathbf{u}_1$ of the domain.
- Because 'Fa' is true in **I**, ⟨**v**(a)⟩ ∈ **v**(F), so ⟨$\mathbf{u}_1$⟩ ∈ **v**(F).
- Then **d**[$\mathbf{u}_1$/x] satisfies 'Fx', and since **d**[$\mathbf{u}_1$/x] satisfies 'Fx ⊃ Gx,' **d**[$\mathbf{u}_1$/x] satisfies 'Gx.'
- It follows that ⟨$\mathbf{u}_1$⟩ ∈ **v**(G), so ⟨**v**(a)⟩ ∈ **v**(G).
- Then **d** satisfies 'Ga,' which is thus true in **I**, QED.

The Example Using Substitutional Semantics

- To prove: {(∀x)(Fx ⊃ Gx), Fa} ⊨ Ga
- Suppose that '(∀x)(Fx ⊃ Gx)' and 'Fa' are true in an interpretation **I**.
- Then all substitution instances of '(∀x)(Fx ⊃ Gx)' are true in **I**.
- Therefore, 'Fa ⊃ Ga' is true in **I**.
- Since 'Fa' and 'Fa ⊃ Ga' are true in **I**, 'Ga' is true in **I**, QED.