Meta-Logic of Predicate Logic

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Validity of Arguments

- An argument of Predicate Logic **Z**, therefore **X** (or **Z\X**) is **valid** if and only if in every interpretation in which all the members of **Z** are true, **X** is also true.
- {(∀x)(Fx ⊃ Gx), Fa}\Ga is a valid argument.
- The following is a proof of validity using Teller's substitutional semantics.
  - Suppose for an arbitrary interpretation **I**, that '(∀x)(Fx ⊃ Gx)' and 'Fa' are true in **I**.
  - Then all substitution instances of '(∀x)(Fx ⊃ Gx)' are true in **I**.
  - Therefore, 'Fa ⊃ Ga' is true in **I**.
  - So, if 'Fa' is true in **I**, then 'Ga' is true in **I**.
  - 'Fa' is true in **I**, so 'Ga' is true in **I**, QED.

Derivability

- Derivability is a relation between a set of sentences **Z** and a sentence **X** on the basis of a given set of rules of inference.
- There are many different sets of rules that yield the same derivability relations, and some which yield different ones.
- Here we will be using the rules of *A Modern Formal Logic Primer*.
- **X** is **derivable** from premises **Z** by Teller's rules if and only if there is a proof of **X** from only premises **Z** using those rules of inference.

1	(∀x)(Fx ⊃ Gx)	P
2	Fa	P
3	Fa ⊃ Ga	1 ∀ E
4	Ga	2 3 ⊃ E

Soundness

- We will use the subscript 'd' after the turnstyle to indicate a derivation according to Teller's rules and the subscript 's' after the double turnstyle to indicate validity in Teller's substitutional semantics.
- $\mathbf{Z} \vdash_{\mathrm{d}} \mathbf{X}$ if and only if **X** is derivable from **Z** according to Teller's rules of inference.
  - '$\vdash$' is a metalogical symbol known as the 'turnstyle.'
  - Without the subscript, it indicates a purely syntactical relation of proof between the members of **Z** and **X** for some system of rules (and possibly axioms).
- $\mathbf{Z} \vDash_{\mathrm{s}} \mathbf{X}$ if and only if the argument **Z\X** is valid.
  - '$\vDash$' is a metalogical symbol known as the 'double turnstyle'.
  - With the subscript, it indicates a semantical relation between the members of **Z** and **X**, according to some system of semantics.
- A system of formal proof is **sound** if and only if for all **Z** and all **X**, if $\mathbf{Z} \vdash \mathbf{X}$, then $\mathbf{Z} \vDash \mathbf{X}$.

Proving Soundness

- To prove soundness, it must be shown that every derivation is such that if all its premises are true in an arbitrary interpretation, then its last step (with no undischarged assumptions) is true as well.
- We have made a start of a proof in our sketches of the soundness of the quantifier rules of inference.
- That is, we have shown in outline how every application of one of the rules of inference does not lead from truth to falsehood.
- A general proof of soundness must show why it is that no **combination** of uses of rules of inference can lead from true premises to a false conclusion.
- To show this, we must use **mathematical induction** (Chapter 11), which is a formal proof procedure in the metalogic that allows us to make inferences to generalized conclusions.

Completeness

- A system of formal proof is **complete** if and only if for all **Z** and all **X**, if $\mathbf{Z} \vDash \mathbf{X}$, then $\mathbf{Z} \vdash \mathbf{X}$ (proved by Kurt Gödel in 1930).
- The proof of completeness is much more difficult than that of soundness.

- Ideally, we would like to be able to show how, for any valid argument, to **construct** a derivation with the same premises and conclusion.
- In practice, logicians give **non-constructive** completeness proofs, which simply show that there is a derivation, without specifying what such a derivation might be.
- Moreover, it is more difficult to construct completeness proofs for derivational systems than for more compact axiomatic systems.
- The standard technique for proving completeness is due to Leon Henkin (1949).

Undecidability

- Given that a system of formal proof is complete, there is a proof for every valid argument.
- In Predicate Logic, there is no purely syntactical way to determine whether a proof for any given argument exists or semantical way to determine whether any given argument is valid.
- There is no mechanical **decision procedure** for determining derivability or validity in a finite number of steps.
- Thus Predicate Logic is **undecidable** (proved by Alonzo Church in 1936).
- However, various fragments of Predicate Logic are decidable.
  - For example, the set of sentences of Predicate Logic in which no predicate has more than one argument.

Other Properties of Predicate Logic

- Besides soundness, completeness and undecidability, there are some other notable properties of modal logic.
- The **Löwenheim-Skolem Theorem** (1919) states that if a sentence **X** is true in any interpretation at all, it is true in some interpretation whose domain is denumerable.
  - A set is **denumerable** if and only if it can be placed in a one-to-one correspondence with the set of natural numbers: $\{0, 1, 2, \ldots\}$.
- The **Compactness Theorem** (Gödel, 1930) states that for any set **Z** of sentences of Predicate Logic, if each finite subset of **Z** is true in at least one interpretation, then there is an interpretation in which all the sentences in **Z** are true.

Second-Order Predicate Logic

- Predicate Logic, originally proposed by Frege, can be extended by allowing quantification over predicates.
  - For example, we might formulate an axiom of identity.
  - $(\forall x)(\forall y)(x = y \equiv (\forall P)(Px \equiv Py))$.
- Semantically, the quantifiers would range over properties or sets, rather than the objects in the domain over which first-order quantifiers range.
- Second-order logic cannot be given a proof system which is sound and complete relative to its interpretation, though fragments of it can be given a sound and complete proof system.
- As with first-order logic, the fragment of second-order logic consisting of none but one-place predicates is decidable.
- It is possible to construct logics of higher, even infinite, order.