

Logical Truth, Contradictions, Inconsistency, and Logical Equivalence

G. J. Matthey

Winter, 2009 / Philosophy 112

Logical Truth

- The semantical concepts of logical truth, contradiction, inconsistency, and logical equivalence in Predicate Logic are straightforward adaptations of the corresponding concepts in Sentence Logic.
- Since only closed sentences of Predicate Logic have truth-values, they are the only sentences that can be logically true, contradictory, etc.
- A closed sentence **X** of Predicate Logic is **logically true**, $\models \mathbf{X}$, if and only if **X** is true in all interpretations.
- The logical truth of a sentence is proved directly using general reasoning in semantics.
- To prove that a sentence is not logically true, one must provide an interpretation in which it is false.

An Example of a Logical Truth of Predicate Logic

- $\models (\forall x)Fx \supset (\exists x)Fx$.
 - Suppose that a variable assignment **d** satisfies ‘ $(\forall x)Fx$.’
 - Then all x-variants of **d** satisfy ‘ Fx ’.
 - Since the domain D is non-empty, some x-variant of **d** satisfies ‘ Fx .’
 - So **d** satisfies ‘ $(\exists x)Fx$.’
 - Therefore **d** satisfies ‘ $(\forall x)Fx \supset (\exists x)Fx$,’ QED.

Theoremhood

- Corresponding to the logical truth of a sentence **X** is its derivation from no premises.
- The result of the derivation is that **X** is **theorem**, $\vdash \mathbf{X}$.

- By **soundness**, if $\vdash X$, then $\models X$.
- By **completeness**, if $\models X$, then $\vdash X$.
- The following is the proof of theoremhood: from $\vdash (\forall x)Fx \supset (\exists x)Fx$ it follows that $\models (\forall x)Fx \supset (\exists x)Fx$.

| | | |
|---|---|-----------------|
| 1 | ($\forall x$)Fx | P |
| 2 | Fa | 1 \forall E |
| 3 | ($\exists x$)Fx | 1 \exists I |
| 4 | ($\forall x$)Fx \supset ($\exists x$)Fx | 1-3 \supset I |

Contradictions

- A closed sentence **X** of Predicate Logic is a **contradiction** if and only if **X** is false in all interpretations.
- A sentence **X** is false in all interpretations if and only if its negation $\sim X$ is true on all interpretations.
- Therefore, one may directly demonstrate that a sentence is a contradiction by proving that its negation is a logical truth.
- If the negation $\sim X$ of a sentence **X** is a logical truth, then given completeness, $\sim X$ is a theorem, and hence $\sim X$ can be derived from no premises.
- If a sentence **X** is such that if it is true in any interpretation, then both **Y** and $\sim Y$ are true in that interpretation, then **X** cannot be true on any interpretation.
- Given soundness, it follows that if **Y** and $\sim Y$ are derivable from **X**, then **X** is a contradiction.

An Example of a Contradiction

- ' $(\forall x)(Fx \ \& \ \sim Fx)$ ' is a contradiction.
 - Suppose that a variable assignment **d** satisfies ' $(\forall x)(Fx \ \& \ \sim Fx)$.'
 - Then all x-variants **d[u/x]** of **d** satisfy ' $Fx \ \& \ \sim Fx$.'
 - Then **d[u/x]** satisfies ' Fx .'
 - Then **d[u/x]** satisfies ' $\sim Fx$.'
 - Then **d[u/x]** does not satisfy ' Fx ,' a contradiction.
 - Therefore, no variable assignment **d** satisfies ' $(\forall x)(Fx \ \& \ \sim Fx)$,' QED.

| | | |
|---|------------------------------|---------------|
| 1 | $(\forall x)(Fx \& \sim Fx)$ | P |
| 2 | $Fa \& \sim Fa$ | 1 \forall E |
| 3 | Fa | 2 & E |
| 4 | $\sim Fa$ | 2 & E |

Inconsistent Sets of Sentences

- A set of closed sentences of Predicate Logic is **consistent** if and only if there is an interpretation (a **model**) which makes all the sentences in the set true.
- A set of closed sentences of Predicate Logic is **inconsistent** just in case it is not consistent.
- Therefore, a set of closed sentences of Predicate Logic is inconsistent just in case it has no models.
- It follows from these definitions and that of a contradiction that a finite collection of sentences is inconsistent if and only if the conjunction of the sentences is a contradiction.
 - There is no model for a set of sentences X if and only if in every interpretation, each of the sentences of X is false.
 - This holds if and only if in every interpretation, the conjunction of the sentences of X is false.
 - This holds if and only if the conjunction of the sentences of X is a contradiction, QED.

Demonstrating Inconsistency

- The consistency of a set of closed sentences can be demonstrated by providing a single interpretation which makes all the sentences in the set true.
- A semantical demonstration of inconsistency requires general reasoning, showing why there is no interpretation in which all the sentences are true.
- Given soundness, inconsistency can be proved by way of derivation be either of two ways.
 - Derive a contradiction from the set of inconsistent sentences taken as premises.
 - Derive the negation of the conjunction of the sentences from no premises.

An Example of Demonstrating Inconsistency Semantically

- Proof that $\{ '(\forall x)Fx', ' \sim(\exists x)Fx' \}$ is inconsistent.
 - Suppose there is an interpretation **I** which makes both $'(\forall x)Fx'$ and $' \sim(\exists x)Fx'$ true.
 - Then for a given variable assignment **d**, **d** satisfies both $'(\forall x)Fx'$ and $' \sim(\exists x)Fx'$.
 - Therefore, all x-variants **d[u/x]** of **d** satisfy $'Fx'$.
 - Since the domain of **I** contains at least one object **v**, **d[v/x]** satisfies $'Fx'$.
 - So **d** satisfies $'(\exists x)Fx'$.
 - Since $' \sim(\exists x)Fx'$ is assumed to be true, **d** satisfies $' \sim(\exists x)Fx'$, which yields a contradiction.
 - So, there is no interpretation which makes both $'(\forall x)Fx'$ and $' \sim(\exists x)Fx'$ true, QED.

An Example of Demonstrating Inconsistency by Derivation

| | | |
|---|----------------------|---------------|
| 1 | $(\forall x)Fx$ | P |
| 2 | $\sim (\exists x)Fx$ | P |
| 3 | Fx | $2 \forall E$ |
| 4 | $(\exists x)Fx$ | $3 \exists I$ |

Logical Equivalence

- Two closed sentences of Predicate Logic are **logically equivalent** if and only if they have the same truth value in all interpretations.
- The logical equivalence of **X** and **Y** holds as well when **X** is true in all interpretations where **Y** is true, and **Y** is true in all interpretations where **X** is true.
- Alternatively, two sentences **X** and **Y** are logically equivalent just in case the biconditional $\mathbf{X} \equiv \mathbf{Y}$ is a logical truth.
- Logical equivalence is demonstrated semantically through general reasoning.
- It is proved, given soundness, with two derivations, each having one of the sentences as a premise and the other as a conclusion.

An Example of Proving Equivalence Semantically

- Proof that $'(\forall x)\sim Fx'$ and $' \sim(\exists x)Fx'$ are logically equivalent.

- $(\forall x)\sim Fx$ is true in I if and only if $(\forall x)\sim Fx$ is satisfied by all variable assignments \mathbf{d} .

$(\forall x)Fx$ is satisfied by all variable assignments \mathbf{d} if and only if $\sim Fx$ is satisfied by all x-variants $\mathbf{d}[u/x]$ of \mathbf{d} .

- $\sim Fx$ is satisfied by all x-variants $\mathbf{d}[u/x]$ of \mathbf{d} if and only if Fx is satisfied by no x-variant $\mathbf{d}[u/x]$ of \mathbf{d} .
- Fx is satisfied by no x-variant $\mathbf{d}[u/x]$ of \mathbf{d} if and only if $(\exists x)Fx$ is not satisfied by \mathbf{d} .
- $(\exists x)Fx$ is not satisfied by \mathbf{d} if and only if $\sim(\exists x)Fx$ is satisfied by \mathbf{d} .
- $\sim(\exists x)Fx$ is satisfied by \mathbf{d} if and only if $(\forall x)\sim Fx$ is true in I, QED.

An Example of Demonstrating Equivalence by Derivation

| | | |
|---|-----------------------|-------------------|
| 1 | $(\forall x) \sim Fx$ | P |
| 2 | $(\exists x)Fx$ | A |
| 3 | x Fa | A |
| 4 | $\sim Fa$ | 1 $\forall E$ |
| 5 | \perp | 3, 4 $\perp I$ |
| 6 | \perp | 2 3-5 $\exists E$ |
| 7 | $\sim (\exists x)Fx$ | 2-6 Reductio |

Demonstrating Equivalence by Derivation, Continued

| | | |
|---|-----------------------|---------------|
| 1 | $\sim (\exists x)Fx$ | P |
| 2 | Fa | A |
| 3 | $(\exists x)Fx$ | 2 $\exists I$ |
| 4 | $\sim (\exists x)Fx$ | 1 R |
| 5 | $\sim Fa$ | 2-4 $\sim I$ |
| 6 | $(\forall x) \sim Fx$ | 5 $\forall I$ |