Introduction to Quantifiers

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Universal Sentences

- Many sentences of natural language make assertions about whole classes of individuals.
- Some of these sentences were called by Aristotle **universal** sentences, though we will call them all "universal."
  - Everyone loves Adam.
- Universal sentences begin with a quantity term ('all,' 'every,' 'any,' 'everybody,' etc.) which may only be implicit, as in the following example.
  - Horses are mammals.
- We would like to be able to symbolize universal sentences, because they play an important role in inference.
  - 1. Everyone loves Adam.
    2. Therefore, Eve loves Adam.

The Syntax of Universal Sentences

- Many universal sentences have a quantity term in the subject position of the sentence.
  - Everyone loves Adam.
- Other universal sentences have quantity term modifying a general term in the subject position of the sentence.
  - All horses are mammals.
- Still other universal sentences do not display the quantity term at all.
  - Horses are mammals.
  - A horse is a mammal.

The Semantics of Universal Sentences

- Semantically, universal quantity terms do not play the role either of subjects or of predicates.
  - They do not designate a single individual, as does a subject of a sentence.
  - They do not say anything about individuals or sets of individuals, as does the predicate of a sentence.
- Instead, universal quantity terms designate the class of all individuals.
- The sentence to which they apply says something about all the members of that class.

Displaying the Behavior of Universal Sentences

- The semantical behavior of the quantity term in the subject position is best brought out in the following formulation.
- Everything is such that it [satisfies the condition stated by the rest of the sentence].
  - Everything is such that it loves Adam.
- The semantical behavior of the quantity term modifying a general term in the subject position can be formulated this way.
- Everything is such that if it [falls under the general term], then it [satisfies the condition stated by the rest of the sentence].
  - Everything is such that if it is a horse, then it is a mammal.

The Universal Quantifier

- In Predicate Logic, the role of 'every' in 'everything' is played by the **universal symbol**, '∀.'
- The role of 'thing' in 'everything' is played by a **variable**, 'w,' 'x,' 'y,' 'z' (with or without positive integer subscripts).
- The whole expression 'everything is such that' combines the universal symbol with a variable, as in '(∀x).'
- This expression of Predicate Logic is called the **universal quantifier**.

Cross-Reference

- The formulation of a universal sentence in English uses ‘it’ to establish **cross-reference** between the quantifier and the rest of the sentence.
- This can be expressed in quasi-English as ‘Every x is such that x [satisfies the condition stated by the rest of the sentence].’
- To establish cross-reference in Predicate Logic, we must put variables in the position taken by constant terms.
  - Eve loves Adam: Lea
  - x loves Adam: Lxa
- An n-place predicate followed by any combination of n constant terms or variables is a sentence of Predicate Logic.
- This explains the way predicates are represented in transcriptions as a predicate followed by n variables.

Transcribing Universal Sentences

- Now we are in a position to display the link between the universal quantifier and the expression containing the variable, first with the quantity term in the subject position.
  - Everyone loves Adam.
  - Every x is such that x loves Adam.
  - (∀x)Lxa, where D = {All people}, a: Adam, Lxy: x loves y.
- Now with the quantity term modifying a general term.
  - Every horse is a mammal.
  - Every x is such that if x is a horse, then x is a mammal.
  - (∀x)(Hx ⊃ Mx), where D = {All things}, Hx: x is a horse, and Mx: x is a mammal.

Governing and Binding

- The boldface Roman lowercase letters ‘**u**’ and ‘**v**’ are metavariables which are to designate variables.
- A universal quantifier (∀**u**) **governs** the shortest full sentence **P**(**u**) following it (as may be marked with parentheses).
  - In the sentence ‘(∀x)(Hx ⊃ Mx),’ ‘(∀x)’ governs the sentence ‘Hx ⊃ Mx.’
  - In the sentence ‘(∀x)Hx ⊃ Mx,’ ‘(∀x)’ governs the sentence ‘Hx.’

- The quantifier **binds** all the occurrences in the sentence it governs of the variable it contains.
  - In the sentence '(∀x)(Hx ⊃ Mx),' '(∀x)' binds both occurrences of 'x' in 'Hx ⊃ Mx.'
  - In the sentence '(∀x)Hx ⊃ Mx,' '(∀x)' binds the occurrence of 'x' in 'Hx.'

Free Variables and Open Sentences

- A variable is **free** in a sentence when it is not bound by any quantifier in that sentence.
  - In the sentence '(∀x)(Hx ⊃ My),' 'x' is bound and 'y' is free.
- A sentence of Predicate Logic which contains at least one free variable is an **open sentence**.
- Some logicians do not consider "open sentences" to be sentences, because they contain terms (variables) which have no intended reference.
  - The sentence '(∀x)(Hx ⊃ My)' would be transcribed into quasi-English as: Everything x is such that if x is a horse, then y is a mammal.
- We count open sentences as sentences for simplicity.

Vacuous Quantification

- The universal quantifier is an operator that creates a sentence of Predicate Logic when prefixed to a sentence of Predicate Logic.
- Sometimes prefixing a universal quantifier to a sentence does not bind a variable.
  - (∀y)(Ha ⊃ Mb)
- Such cases are called cases of **vacuous** quantification.
- We will treat vacuous quantifiers semantically as if they were not there at all.

Interpreting Universally Quantified Sentences

- The 'everything' intended to be captured by the universal quantifier is reflected in the domain of an interpretation.
- If in an interpretation the domain consists of two people, Adam and Eve, then they are 'everything' according to that interpretation.
- So for a univerally quantified sentence $(\forall \mathbf{u})\mathbf{P}(\mathbf{u})$ to be true, it is required that every object in the domain meet the condition specified by the open sentence $\mathbf{P}(\mathbf{u})$.

  - ‘(∀x)Lxa’ is true just in case both Adam and Eve meet the condition specified by ‘Lxa.’
  - v(a) = Adam, requires that both ⟨Adam, Adam⟩ and ⟨Eve, Adam⟩ be in v(L) for ‘(∀x)Lxa’ to be true.

	L	x	a
		↓	↓
		⟨Adam,	Adam⟩
	and	⟨Eve,	Adam⟩

Substitution Instances

- Universally quantified sentences that are not components of other sentences can be converted to **substitution instances** by dropping the quantifier and uniformly substituting a constant term for all the occurrences of the variable in the quantifier.
  - ‘(∀x)(Hx ⊃ Mx)’ ⟶ ‘Ha ⊃ Ma.’
- The constant term is called the **instantiating** constant.
- More generally, a substitution instance of $(\forall \mathbf{u})(\ldots\mathbf{u}\ldots)$ is $(\ldots\mathbf{s/u}\ldots)$, where $(\ldots\mathbf{s/u}\ldots)$ is $(\ldots\mathbf{u}\ldots)$ except that all occurrences of **u** are replaced with **s**.
- Manipulation of substitution instances is the most important kind of move in doing Predicate Logic derivations.

Satisfaction

- A problem stated in the text is that open sentences have no truth-values.
- Nonetheless, we would like to say something about what would happen to an open sentence if we were to let its variable stand for a member of the domain.
- We will say that under this condition, the open sentence is **satisfied**.
  - If in an interpretation ‘x’ is assumed to stand for Adam and ⟨Adam⟩ ∈ v(B), then ‘Bx’ is satisfied given that assumption in the interpretation.
- But as yet we have no means to indicate what variables stand for.

Variable Assignments

- We will expand our semantics by introducing, as components of interpretations, **variable assignments** $\mathbf{d}_1, \mathbf{d}_2, \ldots$ whose arguments are variables and whose values are members of the domain of that interpretation.
- For example, in an interpretation whose domain is {Adam, Eve}, then $d_1(x)$ might assign ‘x’ to Adam and $d_2(x)$ might assign ‘x’ to Eve.

- Then we can say that $\mathbf{P}(\mathbf{x})$ is satisfied by $\mathbf{d}_i$ if and only if $\mathbf{d}_i(\mathbf{x})$ meets the condition specified by $\mathbf{P}(\mathbf{x})$.
- If v(B) = {⟨Adam⟩} then $\langle d_1(x)\rangle$ is in v(B), so d_1 satisfies 'Bx.'
- On the other hand, $\langle d_2(x)\rangle$ is not in v(B), so d_2 does not satisfy 'Bx.'

Designation

- Some sentences containing free variables also contain constant terms.
  - Lxa
- Whether this sentence is satisfied by a variable assignment **d** depends both on what member of the domain **d** assigns to 'x' as well as what member of the domain the valuation function **v** assigns to 'a.'
- If we combine these two notions, we can speak of the **designation** made by either **d** or **v** to a variable and to constant terms, respectively.
- We will write '$\mathbf{des}_{\mathbf{v},\mathbf{d}}$' to indicate the designation function that combines a variable assignment with a valuation.
- Note that the valuation function is fixed by the interpretation, so that any differences between designation functions is the result of differences in variable assignments.

Designation and Satisfaction

- The conditions under which a variable assignment **d** satisfies a sentence of Predicate Logic can be spelled out formally, for a given interpretation **I** and its valuation function **v**.
- If **P** is a sentence letter, then **d** satisfies **P** if and only if **v**(**P**) = t.
- If $\mathbf{P}\mathbf{t}_1\mathbf{t}_2 \ldots \mathbf{t}_n$ is an atomic sentence, then **d** satisfies $\mathbf{P}\mathbf{t}_1\mathbf{t}_2 \ldots \mathbf{t}_n$ if and only if $\langle \mathbf{des}_{\mathbf{v},\mathbf{d}}(\mathbf{t}_1), \mathbf{des}_{\mathbf{v},\mathbf{d}}(\mathbf{t}_2), \ldots, \mathbf{des}_{\mathbf{v},\mathbf{d}}(\mathbf{t}_n) \rangle \in \mathbf{v}(\mathbf{P})$.
- For truth-functional connectives, satisfaction works in the same way as assignment of truth-values.
  - **d** satisfies ~**P** if and only if **d** does not satisfy **P**.
  - **d** satisfies **P** & **Q** if and only if **d** satisfies both **P** and **Q**.
  - And similarly for the other connectives.

Truth-Definition for Universally Quantified Sentences

- Let '**d**[**u**/**x**]' indicate a variable assignment just like **d** with the possible exception of the assignment of a member of the domain **u** to **x**.
  - Suppose **d**(x) = Adam.
  - Then **d**[Eve/x](x) = Eve.
- '**d**[**u**/**x**]' is called an **x-variant** of **d**.
- **d** satisfies a universally quantified sentence (∀**x**)**P**(**x**) in an interpretation **I** if and only if **P**(**x**) is satisfied by the **x**-variants of **d** **d**[**u**/**x**] for all **u** in the domain.
- A sentence **P** of Predicate Logic is true in an interpretation **I** if and only if **P** is satisfied by all variable assignments, which can be seen if an arbitrary variable assignment **d** satisfies it.
- It can be proved that every sentence of predicate logic is satisfied by either all variable assignments or no variable assignments, so every (non-open) sentence of predicate logic is either true or false in a given interpretation.

An Example

- For an interpretation I, D = {Adam, Eve}, v(L) = {⟨Adam, Adam⟩, ⟨Eve, Adam⟩}, v(a) = Adam.
- d[Adam/x] satisfies 'Lxa.'
  - ⟨d[Adam/x](x), v(a)⟩ ∈ v(L).
  - ⟨$\text{des}_{\text{d,v}}$[Adam/x](x), $\text{des}_{\text{d,v}}$(a)⟩ ∈ v(L).
- d[Eve/x] satisfies 'Lxa.'
  - ⟨d[Eve/x](x), v(a)⟩ ∈ v(L).
  - ⟨$\text{des}_{\text{d,v}}$[Eve/x](x), $\text{des}_{\text{d,v}}$(a)⟩ ∈ v(L).
- So, the x-variants of d for all members of the domain satisfy 'Lxa.'
- So, d satisfies '(∀x)Lxa.'
- Since the choice of d is arbitrary, all variable assignments satisfy '(∀x)Lxa,' so the sentence is true in **I**.

Substitutional Semantics for Universally Quantified Sentences

- We have said that for a universally quantified sentence to be true, all members of the domain must satisfy the condition specified by the sentence following the quantifier.
- One way to understand the notion of satisifying the condition specified by the sentence following the quantifier is in terms of the truth of substitution instances of the quantified expression.
  - '(∀x)Lxa' is true if and only if the condition specified by 'Lxa' is satisfied by all members of the domain.
  - Suppose D = {Adam, Eve}, and 'a' designates Adam while 'e' designates Eve.
  - Then the sentence is true if and only if 'Laa' is true and 'Lea' is true.
  - This is because 'Laa' is true if and only if ⟨Adam, Adam⟩ is in the extension of 'L,' and 'Lea' is true if and only if ⟨Eve, Adam⟩ is in the extension of 'L.'

Particular or Existential Sentences

- Many sentences of natural language make assertions about at least one, unspecified, individual.
- Some of these sentences are called by Aristotle **particular** sentences, though we will call them all "particular.'
  - Someone loves Adam.
- Particular sentences begin with an "existential" quantity term (some, there is a(n), there is at least one, there exists).
  - Some horses are mares.
- We would like to be able to symbolize particular sentences, because they play an important role in inference.
  - Eve loves Adam. Therefore, someone loves Adam.

The Syntax of Particular Sentences

- Many particular sentences have a quantity term in the subject position of the sentence.
  - Someone loves Adam.
- Other particular sentences have quantity term modifying a general term in the subject position of the sentence.

  - Some horses are mares.

- Some particular sentences begin with the indefinite article ‘a’ or ‘an.’
  - An alligator is lounging near the pond.

The Semantics of Particular Sentences

- Semantically, existential quantity terms do not play the role either of subjects or of predicates.
  - They do not designate a single individual, as does a subject of a sentence.
  - They do not say anything about individuals or sets of individuals, as does the predicate of a sentence.
- Instead, existential quantity terms designate at least one individual from a class.
- The sentence to which they apply says something about at least one member of the class.

Displaying the Behavior of Particular Sentences

- The semantical behavior of the quantity term in the subject position is best brought out in the following formulation.
- Something is such that it [satisfies the condition stated by the rest of the sentence].
  - Something is such that it is orange.
- The semantical behavior of the quantity term modifying a general term in the subject position can be formulated this way.
- Something is such that it [falls under the general term] and it [satisfies the condition stated by the rest of the sentence].
  - Something is such that it is both a horse and a mare.

The Existential Quantifier

- In Predicate Logic, the role of ‘some’ in ‘something’ is played by the **existential symbol**, ‘∃.’
- The role of ‘thing’ in ‘something’ is played by a variable.
- The whole expression ‘something is such that’ combines the existential symbol with a variable, as in ‘(∃x).’
- This expression of Predicate Logic is called the **existential quantifier**.

Transcribing Particular Sentences

- Now we are in a position to display the link between the existential quantifier and the expression containing the variable, first with the quantity term in the subject position.
  - Someone loves Adam.
  - Some x is such that x loves Adam.
  - (∃x)Lxa, where Lxy: x loves y, a: Adam.
- Now with the quantity term modifying a general term.
  - Some horse is a mare.
  - Some x is such that x is a horse and x is a mare.
  - (∃x)(Hx & Mx), where Hx: x is a horse, and Mx: x is a mare.

Uniform Behavior of Quantifiers

- Much of the terminology applied to universal quantifiers can be applied to existential quantifiers.
- An existential quantifier governs the shortest full sentence following it, and it binds occurrences of its variable in the governed sentence.
- In cases of vacuous quantification, the sentence is interpreted as if the quantifier were not there.
- A substitution instance of an existentially quantified sentence is the sentence governed by the quantifier with all the occurances of the binding variable being replaced by a constant term.

Interpreting Existentially Quantified Sentences

- The 'something' intended to be captured by the existential quantifier is reflected in the domain of an interpretation.
- If in an interpretation the domain consists of two people, Adam and Eve, then either one of the two (or both) is the 'something' according to that interpretation.
- So for an existentially quantified sentence (∃**u**)**P**(**u**) to be true, it is required that at least one object in the domain meet the condition specified by the open sentence **P**(**u**).
  - '(∃x)Lxa' is true just in case either Adam or Eve (inclusively) meet the condition specified by 'Lxa.'
  - Given that 'a' designates Adam, this means that either ⟨Adam, Adam⟩ or ⟨Eve, Adam⟩ is in the extension of 'L.'

L	x	a
	↓	↓
	⟨Adam,	Adam⟩
or	⟨Eve,	Adam⟩

Truth-Definition for Existentially Quantified Sentences

- **d** satisfies an existentially quantified sentence (∃**x**)**P**(**x**) in an interpretation **I** if and only if **P**(**x**) is satisfied by an **x**-variant of **d d**[**u/x**] for some **u** in the domain.
- For an interpretation I, D = {Adam, Eve}, v(L) = {⟨Adam, Adam⟩, ⟨Eve, Adam⟩}, v(a) = Adam.
- d[Eve/x] satisfies 'Lxa.'
- So, an x-variant of d for some member of the domain satisfies 'Lxa.'
- So, d satisfies '(∃x)Lxa.'
- Since the choice of d is arbitrary, all variable assignments satisfy '(∃x)Lxa,' so the sentence is true in **I**.

Substitutional Semantics for Existentially Quantified Sentences

- We have said that for an existentially quantified sentence to be true, at least one member of the domain must satisfy the condition specified by the sentence following the quantifier.
- One way to understand the notion of satisifying the condition specified by the sentence following the quantifier is in terms of the truth of substitution instances of the quantified expression.
  - '(∃x)Lxa' is true if and only if the condition specified by 'Lxa' is satisfied by at least one member of the domain.
  - Suppose D = {Adam, Eve}, and 'a' designates Adam while 'e' designates Eve.
  - Then the sentence is true if and only if 'Laa' is true or 'Lea' is true.
  - This is because 'Laa' is true if and only if ⟨Adam, Adam⟩ is in the extension of 'L,' and 'Lea' is true if and only if ⟨Eve, Adam⟩ is in the extension of 'L.'