Natural Deduction Rules for Quantifiers

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Winter, 2009 / Philosophy 112

Derivation Rules for Quantifiers

- There are two quantifiers in Predicate Logic, each with an introduction rule and an elimination rule.
- The rules vary in difficulty.
  - The simplest rules are Universal Elimination and Existential Introduction.
  - The most difficult rule is Existential Elimination.
  - A rule of intermediate difficulty is Universal Introduction.
- Each rule will be motivated through the semantics for the relevant quantifier.

Universal Elimination

- The rule of Universal Elimination (∀ E, also known as "Universal Instantiation") allows one to remove a universal quantifer and write down a substitution instance of the sentence it governs.

Universal Elimination

m	$(\forall \mathbf{u})\mathbf{P}(\mathbf{u})$	Already Derived
	.	
	.	
	.	
n	$\mathbf{P}(\mathbf{s}/\mathbf{u})$	m ∀E

An Example of Universal Elimination

1	$(\forall x)(Fx \,\&\, Gx)$	A
2	$F\hat{a} \,\&\, G\hat{a}$	1 ∀ E
3	$Ff(\hat{b}) \,\&\, Gf(\hat{b})$	1 ∀ E

Important Features of Universal Elimination

- You may instantiate to either a name or a filled-in function symbol.
- The instantiation must be uniform: there is only one instantiating constant term.
- All occurrences of the variable must be replaced with a constant term.
- It is often essential to instantiate to the proper term, in which case it is best to wait to see what term is required before instantiating.

Sketch of Soundness Proof for Universal Elimination

- Suppose $(\forall \mathbf{u})\mathbf{P}(\mathbf{u})$ is true in an interpretation **I**.
- $(\forall \mathbf{u})\mathbf{P}(\mathbf{u})$ is true in **I** just in case it is satisified by all variable assignments **d** in **I**.
- $(\forall \mathbf{u})\mathbf{P}(\mathbf{u})$ is satisfied by all variable assignments **d** in **I** just in case $\mathbf{P}(\mathbf{u})$ is satisfied by **u**-variants $\mathbf{d}[\mathbf{o}/\mathbf{u}]$ for all **o** in the domain.
- So, $\mathbf{P}(\mathbf{u})$ is satisfied by **u**-variants $\mathbf{d}[\mathbf{o}/\mathbf{u}]$ for all **o** in the domain.
- Since each name **s** designates a member of the domain, $\mathbf{v}(\mathbf{s}) = \mathbf{o}_i$ for some $\mathbf{o}_i$ in the domain.
- It follows that $\mathbf{d}[\mathbf{o}_i/\mathbf{u}]$ satisfies $\mathbf{P}(\mathbf{s}/\mathbf{u})$.
- It can be proved that therefore, **d** satisfies $\mathbf{P}(\mathbf{s}/\mathbf{u})$; as the same argument can be used for any **d**, $\mathbf{P}(\mathbf{s}/\mathbf{u})$ is satisfied by all **d** and hence is true in **I**, QED.

Soundness in Substitutional Semantics

- Proof of soundness given the substitutional semantics is trivial.
- Suppose $(\forall \mathbf{u})\mathbf{P}(\mathbf{u})$ is true in an interpretation **I**.
- $(\forall \mathbf{u})\mathbf{P}(\mathbf{u})$ is true in **I** just in case all substitution instances $\mathbf{P}(\mathbf{s}/\mathbf{u})$ are true in **I**.
- Therefore, $\mathbf{P}(\mathbf{s}/\mathbf{u})$ is true in **I**, QED.

Generalizations

- The rules of Universal and Existential Introduction require a process of **generalization** (the converse of creating substitution instances).
- The generalization of a sentence $\mathbf{P}(\mathbf{s})$ containing a term **s** is obtained by:
  - Deleting all occurrences of **s**,
  - Replacing all these occurrences of **s** with the variable **u**, so that **u** is not bound by any quantifier in the sentence, resulting in $\mathbf{P}(\mathbf{s}/\mathbf{u})$,

  - Prefixing the quantifier to the resulting open sentence to obtain $(\forall \mathbf{u})\mathbf{P}(\mathbf{u})$ or $(\exists \mathbf{u})\mathbf{P}(\mathbf{u})$.

- A universal generalization of 'Fab ∨ Gba' is '(∀x)(Fxb ∨ Gbx)'.
- Existential generalizations of 'R*f*(a)' is '(∃y)Ry' and '(∃z)R*f*(z)'.

Universal Introduction

- The rule of Universal Introduction (∀ I, also known as "Universal Generalization") allows one to replace all occurrences of a name (not a filled-in function symbol) with a variable and prefix a universal quantifier to the beginning of the resulting sentence.

Universal Introduction

m	$\mathbf{P}(\mathbf{s/u})$	Already Derived
	.	
	.	
	.	
n	$(\forall\ \mathbf{u})\mathbf{P}(\mathbf{u})$	m ∀ I

Restrictions on the Use of Universal Introduction

- Unlike Universal Elimination, the rule of Universal Introduction is subject to some special restrictions.
- Generalization may be made only on names, and not on filled-in function symbols.
- The name to be generalized upon must occur **arbitrarily**: it may not appear in any premise or undischarged assumption.
  - The arbitriness of a name is indicated by the circumflex ("hat") over it, as in 'Fâ'.
- One may generalize only on an arbitrary name because the truth of a sentence containing a name occurring in a premise or undischarged assumption can be based on the specific designation of that name.

Arbitrary Occurrences of Names

- The circumflex is written over a name when a use of Universal Introduction is contemplated.
- A circumflex is to be written only if the name occurs at a point where it is not **governed** by any premise or assumption in which the name occurs.
- An occurrence of a name is governed by a premise or assumption just in case that premise or assumption begins at the current scope line or at a scope line continuing to the left of the current scope line.

An Example of Universal Introduction

1	$(\forall x)(Fx \,\&\, Gx)$	A
2	$F\hat{a} \,\&\, G\hat{a}$	1 ∀ E
3	$F\hat{a}$	2 & E
4	$(\forall y)Fy$	3 ∀ I

Sketch of Soundness Proof for Universal Introduction

- We want to move from the truth of **P**(**s/u**) to the truth of (∀**u**)**P**(**u**).
- This is assured only if **P**(**s/u**) is true no matter what the designation of **s** might be.
- Hence, what makes **P**(**s/u**) true must depend on nothing peculiar to the fact that **s** is the designating name; any other name could have done just as well.
- If a name not occurring in a premise or undischarged assumption appears in a sentence of a derivation, it could only do so as the result of the use of Universal Elimination.
- If a name **s** occurs as the result of the use of Universal Elimination, then the truth of **P**(**s/u**) does not depend on anything peculiar to the fact that **s** is the designating name.

Soundness of Universal Introduction in Substitutional Semantics

- In substitutional semantics, we can give the following sketch of a proof of soundness.
- Suppose a substitution instance **P**(**s/u**) of (∀**u**)**P**(**u**) is true in **I**.
- Suppose further that the truth of **P**(**s/u**) is established independently of which member of the domain **s** names.
- Then **P**(**s/u**) is true for all substitution instances of (∀**u**)**P**(**u**).
- Therefore, (∀**u**)**P**(**u**) is true in **I**, QED.

Existential Introduction

- The rule of Existential Introduction (∃ I, also known as "Existential Generalization") allows one to replace any number of occurrences of a constant term (name or filled-in function symbol) with a free variable and prefix an existential quantifier to the beginning of the resulting sentence.

Existential Introduction

m	**P**(**s**/**u**)	Already Derived
	.	
	.	
	.	
n	(∃ **u**)**P**(**u**)	m ∃ I

Two Examples of Existential Introduction

1	(∀x)Fx	A
2	Fa	1 ∀ E
3	(∃y)Fy	2 ∃ I

1	F*g*(a)b	A
2	(∃x)F*g*(x)b	1 ∃ I
3	(∃x)F*g*(a)x	1 ∃ I
4	(∃x)Fxb	1 ∃ I

Sketch of Soundness Proof for Existential Introduction

- Suppose for an arbitrary interpretation **I**, **P**(**s**/**u**) is true in **I**, where v(**s**) = $\mathbf{o}_i$, which is in the domain **D** of **I**.
- It follows that **d**[$\mathbf{o}_i$/**u**] satisfies **P**(**u**).
- Therefore, at least one **u**-variant of **d** satisfies **P**(**u**), in which case **d** satisfies (∃**u**)**P**(**u**).
- It follows that (∃**u**)**P**(**u**) is true in **I**, QED.
- As with Universal Introduction, the proof is trivial in the substitutional semantics.

Existential Elimination

- The rule of Existential Elimination (∃ E, also known as "Existential Instantiation") allows one to remove an existential quantifier, replacing it with a substitution instance, made with an unused name, within a new assumption. A sentence not containing the name is derived from that assumption, and the assumption is discharged, with the sentence brought out.

Existential Elimination

i	$(\exists \mathbf{u})\mathbf{P}(\mathbf{u})$	Existing step
j	s | $\mathbf{P}(\mathbf{s}/\mathbf{u})$	Assumption
	| ...	
k	| $\mathbf{X}$	Derived from earlier steps
l	$\mathbf{X}$	i j-k ∃ E

Restrictions on the Use of Existential Elimination

- The instantiating name must be "isolated" (not occurring in anywhere earlier in the derivation).
- The isolation of the name is indicated by writing it to the left of the scope line next to the assumption.
- The instantiating name must not occur in the sentence **X** which is derived from the substitution instance of the existential sentence.

An Example of Existential Elimination

1	(∃x)Fx	P
2	a | Fa	A
3	| (∃y)Fy	2 ∃ I
4	(∃y)Fy	1 2-3 ∃ E

Another Example of Existential Elimination

1	(∃x)(Fx&Gx)	P
2	a | Fa&Ga	A
3	| Fa	1 & E
4	| Ga	1 & E
5	| (∃y)Fy	3 ∃ I
6	| (∃z)Gz	4 ∃ I
7	| (∃y)Fy&(∃z)Gz	5 6 & I
8	(∃y)Fy&(∃z)Gz	1 2-7 ∃ E

Sketch of Soundness Proof for Existential Elimination

- We want to move from the truth of $(\exists \mathbf{u})\mathbf{P}(\mathbf{u})$ to the truth of a sentence **X** derived from one of its substitution instances.
- When we instantiate an existential sentence $(\exists \mathbf{u})\mathbf{P}(\mathbf{u})$ to $\mathbf{P}(\mathbf{s}/\mathbf{u})$, we use the name **s** to designate an unspecified member of the domain: whatever it is that meets the condition of the open sentence following it.
- Assume that the substitution instance $\mathbf{P}(\mathbf{s}/\mathbf{u})$is true.
- If the truth of a sentence **X** follows from that of the substitution instance, in such a way that the choice of the specific name **s** plays no role whatsoever in making it true, then **X** itself is a true sentence, given the assumption.
- We can dispense with the assumption because of the fact that the choice of name is irrelevant to making **X** true, and we can count **X** as true.
- The same argument can be used in the substitutional semantics, except that the reference to the domain is omitted.