

## Introduction to Quantifiers

### Universal Sentences

- Many sentences of natural language make assertions about whole classes of individuals.
- Some of these sentences were called by Aristotle **universal** sentences, though we will call them all “universal”.
  - Everyone loves Adam.
- Universal sentences begin with a quantity term (‘all’, ‘every’, ‘any’, ‘everybody’, etc.) which may only be implicit.
  - Horses are mammals.
- We would like to be able to symbolize universal sentences, because they play an important role in inference.
  - Everyone loves Adam. Therefore, Eve loves Adam.

### The Syntax of Universal Sentences

- Many universal sentences have a quantity term in the subject position of the sentence.
  - Everyone loves Adam.
- Other universal sentences have quantity term modifying a general term in the subject position of the sentence.
  - All horses are mammals.
- Still other universal sentences do not display the quantity term at all.
  - Horses are mammals.
  - A horse is a mammal.

### The Semantics of Universal Sentences

- Semantically, universal quantity terms do not play the role either of subjects or of predicates.
  - They do not designate a single individual, as does a subject of a sentence.
  - They do not say anything about individuals or sets of individuals, as does the predicate of a sentence.
- Instead, universal quantity terms designate the class of all individuals.
- The sentence to which they apply says something about all the members of that class.

### Displaying the Behavior of Universal Sentences

- The semantical behavior of the quantity term in the subject position is best brought out in the following formulation.
- Everything is such that it [satisfies the condition stated by the rest of the sentence].
  - Everything is such that it loves Adam.
- The semantical behavior of the quantity term modifying a general term in the subject position can be formulated this way.
- Everything is such that if it [falls under the general term], then it [satisfies the condition stated by the rest of the sentence].
  - Everything is such that if it is a horse, then it is a mammal.

### The Universal Quantifier

- In Predicate Logic, the role of ‘every’ in ‘everything’ is played by the **universal symbol**, ‘ $\forall$ ’.
- The role of ‘thing’ in ‘everything’ is played by a **variable**, ‘w’, ‘x’, ‘y’, ‘z’ (with or without positive integer subscripts).
- The whole expression ‘everything is such that’ combines the universal symbol with a variable, as in ‘ $(\forall x)$ ’.
- This expression of Predicate Logic is called the **universal quantifier**.

### Cross-Reference

- The formulation of a universal sentence in English uses ‘it’ to establish **cross-reference** between the quantifier and the rest of the sentence.
- This can be expressed in quasi-English as ‘Every x is such that x [satisfies the condition stated by the rest of the sentence]’.
- To establish cross-reference in Predicate Logic, we must put variables in the position taken by constant terms.
  - Eve loves Adam: Lea
  - x loves Adam: Lxa
- An n-place predicate followed by any combination of n constant terms or variables is a sentence of Predicate Logic.
- This explains the way predicates are represented in transcriptions.

### Transcribing Universal Sentences

- Now we are in a position to display the link between the universal quantifier and the expression containing the variable, first with the quantity term in the subject position.
  - Everyone loves Adam.
  - Every  $x$  is such that  $x$  loves Adam.
  - $(\forall x)Lxa$ , where  $D = \{\text{All people}\}$ ,  $a$ : Adam,  $Lxy$ :  $x$  loves  $y$ .
- Now with the quantity term modifying a general term.
  - Every horse is a mammal.
  - Every  $x$  is such that if  $x$  is a horse, then  $x$  is a mammal.
  - $(\forall x)(Hx \supset Mx)$ , where  $D = \{\text{All things}\}$ ,  $Hx$ :  $x$  is a horse, and  $Mx$ :  $x$  is a mammal.

### Governing and Binding

- The boldface Roman lowercase letters '**u**' and '**v**' are metavariables which are to designate variables.
- A universal quantifier  $(\forall \mathbf{u})$  **governs** the shortest full sentence **P(u)** following it.
  - In the sentence  $(\forall x)(Hx \supset Mx)$ , ' $(\forall x)$ ' governs the sentence ' $Hx \supset Mx$ '.
  - In the sentence  $(\forall x)Hx \supset Mx$ , ' $(\forall x)$ ' governs the sentence ' $Hx$ '.
- The quantifier **binds** all the occurrences in the sentence it governs of the variable it contains.
  - In the sentence  $(\forall x)(Hx \supset Mx)$ , ' $(\forall x)$ ' binds both occurrences of ' $x$ ' in ' $Hx \supset Mx$ '.
  - In the sentence  $(\forall x)Hx \supset Mx$ , ' $(\forall x)$ ' binds the occurrence of ' $x$ ' in ' $Hx$ '.

### Free Variables and Open Sentences

- A variable is **free** in a sentence when it is not bound by any quantifier in that sentence.
  - In the sentence  $(\forall x)(Hx \supset My)$ , ' $x$ ' is bound and ' $y$ ' is free.
- A sentence of Predicate Logic which contains at least one free variable is an **open sentence**.
- Some logicians do not consider "open sentences" to be sentences, because they contain terms (variables) which have no intended reference.
  - The sentence  $(\forall x)(Hx \supset My)$  would be transcribed as: Everything  $x$  is such that if  $x$  is a horse, then  $y$  is a mammal.
- We count open sentences as sentences for simplicity.

### Vacuous Quantification

- The universal quantifier is an operator that creates a sentence of Predicate Logic when prefixed to a sentence of Predicate Logic.
- Sometimes prefixing a universal quantifier to a sentence does not bind a variable.
  - $(\forall y)(Ha \supset Mb)$
- Such cases are called cases of **vacuous** quantification.
- We will treat vacuous quantifiers as if they were not there at all.

### Interpreting Universally Quantified Sentences

- The ‘everything’ intended to be captured by the universal quantifier is reflected in the domain of an interpretation.
- If in an interpretation the domain consists of two people, Adam and Eve, then they are ‘everything’ according to that interpretation.
- So for a universally quantified sentence  $(\forall \mathbf{u})\mathbf{P}(\mathbf{u})$  to be true, it is required that every object in the domain meet the condition specified by the open sentence  $\mathbf{P}(\mathbf{u})$ .
  - ‘ $(\forall x)Lxa$ ’ is true just in case both Adam and Eve meet the condition specified by ‘ $Lxa$ ’.
  - $v(a) = \text{Adam}$ , this means that both  $\langle \text{Adam}, \text{Adam} \rangle$  and  $\langle \text{Eve}, \text{Adam} \rangle$  are in  $v(L)$ .

L	x	a
	↓	↓
	$\langle \text{Adam},$	Adam $\rangle$
and	$\langle \text{Eve},$	Adam $\rangle$

### Substitution Instances

- Universally quantified sentences can be converted to **substitution instances** by dropping the quantifier and uniformly substituting a constant term for all the occurrences of the variable in the quantifier.
  - ‘ $(\forall x)(Hx \supset Mx)$ ’  $\longrightarrow$  ‘ $Ha \supset Ma$ ’.
- The constant term is called the **instantiating** constant.
- More generally, a substitution instance of  $(\forall \mathbf{u})(\dots \mathbf{u} \dots)$  is  $(\dots \mathbf{s}/\mathbf{u} \dots)$ , where  $(\dots \mathbf{s}/\mathbf{u} \dots)$  is  $(\dots \mathbf{u} \dots)$  except that all occurrences of  $\mathbf{u}$  are replaced with  $\mathbf{s}$ .
- Manipulation of substitution instances is the most important kind of move in doing Predicate Logic derivations.

### Satisfaction

- A problem stated in the text is that open sentences have no truth-values.
- Nonetheless, we would like to say something about what would happen to an open sentence if we were to let its variable stand for a member of the domain.
- We will say that under this condition, the open sentence is **satisfied**.
  - If in an interpretation ‘x’ designates Adam and  $\langle \text{Adam} \rangle \in v(B)$ , then ‘Bx’ is satisfied by that designation in the interpretation.
- But as yet we have no means to indicate the designation of variables.

### Designation Functions

- We will expand our semantics by introducing, as components of interpretations, **variable assignments**  $\mathbf{d}_1, \mathbf{d}_2, \dots$  whose arguments are variables and whose values are members of the domain of that interpretation.
- For example, in an interpretation whose domain is {Adam, Eve}, then  $\mathbf{d}_1(x)$  might designate Adam and  $\mathbf{d}_2(x)$  Eve.
- Then we can say that  $\mathbf{P}(x)$  is satisfied by  $\mathbf{d}_i$  if and only if  $\mathbf{d}_i(x)$  meets the condition specified by  $\mathbf{P}(x)$ .
- If  $v(B) = \{\langle \text{Adam} \rangle\}$  then  $\langle \mathbf{d}_1(x) \rangle$  is in  $v(B)$ , so  $\mathbf{d}_1$  satisfies ‘Bx’.
- On the other hand,  $\langle \mathbf{d}_2(x) \rangle$  is not in  $v(B)$ , so  $\mathbf{d}_2$  does not satisfy ‘Bx’.

### Designation and Satisfaction

- The conditions under which a variable assignment  $\mathbf{d}$  satisfies a sentence of Predicate Logic can be spelled out formally, for a given interpretation  $\mathbf{I}$ .
- If  $\mathbf{P}$  is a sentence letter, then  $\mathbf{d}$  satisfies  $\mathbf{P}$  if and only if  $v(\mathbf{P}) = t$ .
- If  $\mathbf{P}t_1t_2 \dots t_n$  is an atomic sentence, then  $\mathbf{d}$  satisfies  $\mathbf{P}t_1t_2 \dots t_n$  if and only if  $\langle \mathbf{d}(t_1), \mathbf{d}(t_2), \dots, \mathbf{d}(t_n) \rangle \in v(\mathbf{P})$ .
- For truth-functional connectives, satisfaction works in the same way as assignment of truth-values.
  - $\mathbf{d}$  satisfies  $\sim\mathbf{P}$  if and only if  $\mathbf{d}$  does not satisfy  $\mathbf{P}$ .
  - $\mathbf{d}$  satisfies  $\mathbf{P} \& \mathbf{Q}$  if and only if  $\mathbf{d}$  satisfies both  $\mathbf{P}$  and  $\mathbf{Q}$ .
  - And similarly for the other connectives.

### Truth-Definition for Universally Quantified Sentences

- Let ' $\mathbf{d}[\mathbf{u}/\mathbf{x}]$ ' indicate a variable assignment just like  $\mathbf{d}$  with the possible exception of the assignment of a member of the domain  $\mathbf{u}$  to  $\mathbf{x}$ .
  - Suppose  $\mathbf{d}(\mathbf{x}) = \text{Adam}$ .
  - Then  $\mathbf{d}[\text{Eve}/\mathbf{x}](\mathbf{x}) = \text{Eve}$ .
- ' $\mathbf{d}[\mathbf{u}/\mathbf{x}]$ ' is called an **x-variant** of  $\mathbf{d}$ .
- $\mathbf{d}$  satisfies a universally quantified sentence  $(\forall \mathbf{x})\mathbf{P}(\mathbf{x})$  in an interpretation  $\mathbf{I}$  if and only if  $\mathbf{P}(\mathbf{x})$  is satisfied by the  $\mathbf{x}$ -variants of  $\mathbf{d}$   $\mathbf{d}[\mathbf{u}/\mathbf{x}]$  for all  $\mathbf{u}$  in the domain.
- A sentence  $\mathbf{P}$  of Predicate Logic is true in an interpretation  $\mathbf{I}$  if and only if  $\mathbf{P}$  is satisfied by all variable assignments, which can be seen if an arbitrary variable assignment  $\mathbf{d}$  satisfies it.

### An Example

- For an interpretation  $\mathbf{I}$ ,  $D = \{\text{Adam}, \text{Eve}\}$ ,  $v(L) = \{\langle \text{Adam}, \text{Adam} \rangle, \langle \text{Eve}, \text{Adam} \rangle\}$ ,  $v(a) = \text{Adam}$ .
- $\langle \mathbf{d}[\text{Adam}/\mathbf{x}](\mathbf{x}), v(a) \rangle$  satisfies ' $Lx a$ '.
- $\langle \mathbf{d}[\text{Eve}/\mathbf{x}](\mathbf{x}), v(a) \rangle$  satisfies ' $Lx a$ '.
- So, the  $\mathbf{x}$ -variants of  $\mathbf{d}$  for all members of the domain satisfy ' $Lx a$ '.
- So,  $\mathbf{d}$  satisfies ' $(\forall \mathbf{x})Lx a$ '.
- Since the choice of  $\mathbf{d}$  is arbitrary, all variable assignments satisfy ' $(\forall \mathbf{x})Lx a$ ', so the sentence is true in  $\mathbf{I}$ .

### Substitutional Semantics for Universally Quantified Sentences

- We have said that for a universally quantified sentence to be true, all members of the domain must satisfy the condition specified by the sentence following the quantifier.
- One way to understand the notion of satisfying the condition specified by the sentence following the quantifier is in terms of the truth of substitution instances of the quantified expression.
  - ' $(\forall \mathbf{x})Lx a$ ' is true if and only if the condition specified by ' $Lx a$ ' is satisfied by all members of the domain.
  - Suppose  $D = \{\text{Adam}, \text{Eve}\}$ , and ' $a$ ' designates Adam while ' $e$ ' designates Eve.
  - Then the sentence is true if and only if ' $Laa$ ' is true and ' $Lea$ ' is true.
  - This is because ' $Laa$ ' is true if and only if  $\langle \text{Adam}, \text{Adam} \rangle$  is in the extension of ' $L$ ', and ' $Lea$ ' is true if and only if  $\langle \text{Eve}, \text{Adam} \rangle$  is in the extension of ' $L$ '.

### **Particular or Existential Sentences**

- Many sentences of natural language make assertions about at least one, unspecified, individual.
- Some of these sentences are called by Aristotle **particular** sentences, though we will call them all “particular”.
  - Someone loves Adam.
- Particular sentences begin with an “existential” quantity term (some, there is a(n), there is at least one, there exists).
  - Some horses are mares.
- We would like to be able to symbolize particular sentences, because they play an important role in inference.
  - Eve loves Adam. Therefore, someone loves Adam.

### **The Syntax of Particular Sentences**

- Many particular sentences have a quantity term in the subject position of the sentence.
  - Someone loves Adam.
- Other particular sentences have quantity term modifying a general term in the subject position of the sentence.
  - Some horses are mares.
- Some particular sentences begin with the indefinite article ‘a’ or ‘an’.
  - An alligator is lounging near the pond.

### **The Semantics of Particular Sentences**

- Semantically, existential quantity terms do not play the role either of subjects or of predicates.
  - They do not designate a single individual, as does a subject of a sentence.
  - They do not say anything about individuals or sets of individuals, as does the predicate of a sentence.
- Instead, existential quantity terms designate at least one individual from a class.
- The sentence to which they apply says something about at least one member of the class.

### Displaying the Behavior of Particular Sentences

- The semantical behavior of the quantity term in the subject position is best brought out in the following formulation.
- Something is such that it [satisfies the condition stated by the rest of the sentence].
  - Something is such that it is orange.
- The semantical behavior of the quantity term modifying a general term in the subject position can be formulated this way.
- Something is such that it [falls under the general term] and it [satisfies the condition stated by the rest of the sentence].
  - Something is such that it is both a horse and a mare.

### The Existential Quantifier

- In Predicate Logic, the role of ‘some’ in ‘something’ is played by the **existential symbol**, ‘ $\exists$ ’.
- The role of ‘thing’ in ‘something’ is played by a variable.
- The whole expression ‘something is such that’ combines the universal symbol with a variable, as in ‘ $(\exists x)$ ’.
- This expression of Predicate Logic is called the **existential quantifier**.

### Transcribing Particular Sentences

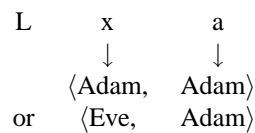
- Now we are in a position to display the link between the existential quantifier and the expression containing the variable, first with the quantity term in the subject position.
  - Someone loves Adam.
  - Some  $x$  is such that  $x$  loves Adam.
  - $(\exists x)Lxa$ , where  $Lxy$ :  $x$  loves  $y$ ,  $a$ : Adam.
- Now with the quantity term modifying a general term.
  - Some horse is a mare.
  - Some  $x$  is such that  $x$  is a horse and  $x$  is a mammal.
  - $(\exists x)(Hx \ \& \ Mx)$ , where  $Hx$ :  $x$  is a horse, and  $Mx$ :  $x$  is a mare.

### Uniform Behavior of Quantifiers

- Much of the terminology applied to universal quantifiers can be applied to existential quantifiers.
- An existential quantifier governs the shortest full sentence following it, and it binds occurrences of its variable in the governed sentence.
- In cases of vacuous quantification, the sentence is interpreted as if the quantifier were not there.
- A substitution instance of an existentially quantified sentence is the sentence governed by the quantifier with all the occurrences of the binding variable being replaced by a constant term.

### Interpreting Existentially Quantified Sentences

- The ‘something’ intended to be captured by the existential quantifier is reflected in the domain of an interpretation.
- If in an interpretation the domain consists of two people, Adam and Eve, then one of the two is ‘something’ according to that interpretation.
- So for an existentially quantified sentence  $(\exists \mathbf{u})\mathbf{P}(\mathbf{u})$  to be true, it is required that at least one object in the domain meet the condition specified by the open sentence  $\mathbf{P}(\mathbf{u})$ .
  - ‘ $(\exists x)Lxa$ ’ is true just in case either Adam or Eve (inclusively) meet the condition specified by ‘ $Lxa$ ’.
  - Given that ‘a’ designates Adam, this means that either  $\langle \text{Adam}, \text{Adam} \rangle$  or  $\langle \text{Eve}, \text{Adam} \rangle$  is in the extension of ‘L’.



### Truth-Definition for Existentially Quantified Sentences

- $\mathbf{d}$  satisfies an existentially quantified sentence  $(\exists \mathbf{x})\mathbf{P}(\mathbf{x})$  in an interpretation  $\mathbf{I}$  if and only if  $\mathbf{P}(\mathbf{x})$  is satisfied by an  $\mathbf{x}$ -variant of  $\mathbf{d}$   $\mathbf{d}[\mathbf{u}/\mathbf{x}]$  for some  $\mathbf{u}$  in the domain.
- For an interpretation  $\mathbf{I}$ ,  $D = \{ \text{Adam}, \text{Eve} \}$ ,  $v(L) = \{ \langle \text{Adam}, \text{Adam} \rangle, \langle \text{Eve}, \text{Adam} \rangle \}$ ,  $v(a) = \text{Adam}$ .
- $\langle \mathbf{d}[\text{Eve}/x](x), v(a) \rangle$  satisfies ‘ $Lxa$ ’.
- So, an  $x$ -variant of  $\mathbf{d}$  for some member of the domain satisfies ‘ $Lxa$ ’.
- So,  $\mathbf{d}$  satisfies ‘ $(\exists x)Lxa$ ’.
- Since the choice of  $\mathbf{d}$  is arbitrary, all variable assignments satisfy ‘ $(\exists x)Lxa$ ’, so the sentence is true in  $\mathbf{I}$ .

### Substitutional Semantics for Existentially Quantified Sentences

- We have said that for an existentially quantified sentence to be true, at least one member of the domain must satisfy the condition specified by the sentence following the quantifier.
- One way to understand the notion of satisfying the condition specified by the sentence following the quantifier is in terms of the truth of substitution instances of the quantified expression.
  - ‘ $(\exists x)Lxa$ ’ is true if and only if the condition specified by ‘ $Lxa$ ’ is satisfied by at least one member of the domain.
  - Suppose  $D = \{\text{Adam, Eve}\}$ , and ‘a’ designates Adam while ‘e’ designates Eve.
  - Then the sentence is true if and only if ‘ $Laa$ ’ is true or ‘ $Lea$ ’ is true.
  - This is because ‘ $Laa$ ’ is true if and only if  $\langle \text{Adam, Adam} \rangle$  is in the extension of ‘L’, and ‘ $Lea$ ’ is true if and only if  $\langle \text{Eve, Adam} \rangle$  is in the extension of ‘L’.